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Special holonomy sigma models with boundaries

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Abstract: A study of $(1,1)$ supersymmetric two-dimensional non-linear sigma models with boundary on special holonomy target spaces is presented. In particular, the consistency of the boundary conditions under the various symmetries is studied. Models both with and without torsion are discussed.

Keywords: Supersymmetric Effective Theories, Global Symmetries, Sigma Models.

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Contents

1. Introduction

There has been a long history of interplay between differential geometry and supersymmetric non-linear sigma models starting with the observation that $N = 2$ supersymmtery in two dimensions requires the sigma model target space to be a Kähler manifold [1]. It was first pointed out in [2] that one could construct conserved currents in $(1, 1)$ sigma models given a covariantly constant form on the target space, and in [3] it was shown that the $(1, 1)$ model on a Calabi-Yau three-fold has an extended superconformal algebra involving precisely such a current constructed from the holomorphic three-form. In [4] symmetries of this type were studied systematically in the classical sigma model setting; each manifold on Berger's list of irreducible non-symmetric Riemannian manifolds has one or more covariantly constant forms which give rise to conserved currents and the corresponding Poisson bracket algebras are non-linear, i.e. they are of W-symmetry type. Subsequently the properties of these algebras were studied more abstractly in a conformal field theory framework [5, 6] and more recently in topological models [7].

In this paper we shall discuss two-dimensional (1,1) supersymmetric sigma models with boundaries with extra symmetries of the above type, focusing in particular on target spaces with special holonomy. In a series of papers [8]–[12] classical supersymmetric sigma models with boundaries have been discussed in detail and it has been shown how the fermionic boundary conditions involve a locally defined tensor R which determines the geometry associated with the boundary. In particular, in the absence of torsion, one finds that there are integral submanifolds of the projector $P = \frac{1}{2}(1 + R)$ which have the interpretation of being branes where the boundary can be located. These papers considered $(1, 1)$ and

$(2,2)$ models and the analysis was also extended to models of this type with torsion where the intepretation of R is less straightforward. The main purpose of the current paper is to further extend this analysis to include symmetries associated with certain holonomy groups or G-structures. We shall discuss models both with and without torsion.

Torsion-free sigma models with boundaries on manifolds with special holonomy were first considered in [13] where it was shown how the identification of the left and right currents on the boundary has a natural interpretation in terms of calibrations and calibrated submanifolds. Branes have also been discussed extensively in boundary CFT [14], including the G_2 case [15], and in topological string theory [16].

The main new results of the paper concern boundary (1,1) models with torsion or with a gauge field on the brane. There is no analogue of Berger's list in the case of torsion but we can nevertheless consider target spaces with specific G-structures which arise due to the presence of covariantly constant forms of the same type. In order to generalise the discussion from the torsion-free case we require there to be two independent G-structures specified by two sets of covariantly constant forms $\{\lambda^+, \lambda^-\}$ which are covariantly constant with respect to two metric connections $\{\Gamma^+, \Gamma^-\}$ and which have closed skew-symmetric torsion tensors $T^{\pm} = \pm H$, where $H = db$, b being the two-form potential which appears in the sigma model action. This sort of structure naturally generalises the notion of bi-hermitian geometry which occurs in $N = 2$ sigma models with torsion [17, 4] and which has been studied in the boundary sigma model context in [10]. We shall refer to this type of structure as a bi-G-structure. The groups G which are of most interest from the point of view of spacetime symmetry are the groups which appear on Berger's list and for this reason we use the term special holonomy. Bi-G-structures are closely related to the generalised structures which have appeared in the mathematical literature [19 – 21]. These generalised geometries have been discussed in the $N = 2$ sigma model context [22 – 24]. In a recent paper they have been exploited in the context of branes and generalised calibrations.

We shall show that, in general, the geometrical conditions implied by equating the left and right currents on the boundary lead to further constraints by differentiation and that these constraints are the same as those which arise when one looks at the stability of the boundary conditions under symmetry transformations. It turns out, however, that these constraints are automatically satisfied by virtue of the target space geometry.

We then study the target space geometry of some examples, in particular bi-G_2, bi-$SU(3)$ and bi-$Spin(7)$ structures. Structures of this type have appeared in the supergravity literature in the context of supersymmetric solutions with flux [27, 28, 30].

The paper is organised as follows: in section 2 we review the basics of boundary sigma models, in section 3 we discuss additional symmetries associated with special holonomy groups or bi-G-structures, in section 4 we examine the consistency of the boundary conditions under symmetry variations, in section 5 we look at the target space geometry of bi-G structures from a simple point of view and in section 6 we look at some examples of solutions of the boundary conditions for the currents defined by the covariantly constant forms.

2. Review of basics

The action for a $(1,1)$-supersymmetric sigma model without boundary is

$$S = \int dz\, e_{ij} D_+ X^i D_- X^j \ , \tag{2.1}$$

where

$$e_{ij} := g_{ij} + b_{ij} \ , \tag{2.2}$$

b being a two-form potential with field strength $H = db$ on the n-dimensional Riemannian target space (M, g). $X^i, i = 1, \dots n$, is the sigma model field represented in some local chart for M and z denotes the coordinates of $(1,1)$ superspace Σ. We shall use a light-cone basis so that $z = (x^{++}, x^{--}, \theta^+, \theta^-)$, with $x^{++} = x^0 + x^1, x^{--} = x^0 - x^1$. D_+ and D_- are the usual flat superspace covariant derivatives which obey the relations

$$D_+^2 = i\partial_{++}; \qquad D_-^2 = i\partial_{--}; \qquad \{D_+, D_-\} = 0 \ . \tag{2.3}$$

We use the convention that $\partial_{++} x^{++} = 1$. We shall take the superspace measure to be

$$dz := d^2x\, D_+ D_- \tag{2.4}$$

with the understanding that the superfield obtained after integrating over the odd variables (i.e after applying D_+D_- to the integrand) is to be evaluated at $\theta = 0$.

As well as the usual Levi-Civita connection ∇ there are two natural metric connections $\nabla^{\pm}$ with torsion [17, 18],

$$\Gamma^{(\pm)j}{}_{ik} := \Gamma^j_{ik} \pm \frac{1}{2} H^j{}_{ik} \ . \tag{2.5}$$

The torsion tensors of the two connections are given by

$$T^{(\pm)i}{}_{jk} = \pm H^i{}_{jk} \ , \tag{2.6}$$

so that the torsion is a closed three-form in either case.

In the presence of a boundary, $\partial\Sigma$, it is necessary to add additional boundary terms to the action (2.2) when there is torsion [9]. The boundary action is

$$S_{bdry} = \int_{\partial\Sigma} a_i \dot{X}^i + \frac{i}{4} b_{ij} (\psi_+^i \psi_+^j + \psi_-^i \psi_-^j) \ , \tag{2.7}$$

where a_i is a gauge field which is defined only on the submanifold where the boundary sigma model field maps takes its values. Note that the boundary here is purely bosonic so that the fields are component fields, $\psi_\pm^i := D_\pm X^i|$, the vertical bar denoting the evaluation of a superfield at $\theta = 0$).[1] The boundary term ensures that the action is unchanged if we add dc to b provided that we shift a to $a - c$. The modified field strength $F = f + b$, where $f = da$, is invariant under this transformation. In the absence of a b-field one can still have a gauge field on the boundary.

[1] We shall use X^i to mean either the superfield or its leading component; it should be clear from the context which is meant.

In the following we briefly summarise the approach to boundary sigma models of references [8]–[12]. We impose the standard boundary conditions [31] on the fermions,

$$\psi_-^i = \eta R^i{}_j \psi_+^j, \qquad \eta = \pm 1, \qquad \text{on } \partial\Sigma \tag{2.8}$$

We shall also suppose that there are both Dirichlet and Neumann directions for the bosons. That is, we assume that there is a projection operator Q such that

$$Q^i{}_j \delta X^j = Q^i{}_j \dot{X}^j = 0 \ , \tag{2.9}$$

on $\partial\Sigma$. If $F = 0$, parity implies that $R^2 = 1$, so that $Q = \frac{1}{2}(1 - R)$, while $P := \frac{1}{2}(1 + R)$ is the complementary projector. In general, we shall still use P to denote $\frac{1}{2}(1 + R)$ and the complementary projector will be denoted by π, $\pi := 1 - Q$. We can take Q and π to be orthogonal

$$\pi_i^k g_{kl} Q^l{}_j = 0 \ . \tag{2.10}$$

Equation (2.9) must hold for any variation along the boundary. Making a supersymmetry transformation we find

$$QR + Q = 0 \ . \tag{2.11}$$

On the other hand, the cancellation of the fermionic terms in the boundary variation (of $S + S_{bdry}$), when the bulk equations of motion are satisfied, requires

$$g_{ij} = g_{kl} R^k{}_i R^l{}_j \ . \tag{2.12}$$

Using this together with orthogonality one deduces the following algebraic relations,

$$\begin{aligned} QR &= RQ = -Q; \qquad & QP = PQ = 0; \\ \pi P &= P\pi = P; \qquad & \pi R = R\pi \ . \end{aligned} \tag{2.13}$$

Making a supersymmetry variation of the fermionic boundary condition (2.8) and using the equation of motion for the auxiliary field, $F^i := \nabla_-^{(+)} D_+ X^i|$, namely $F^i = 0$, we find the bosonic boundary condition[2]

$$i(\partial_{--} X^i - R^i{}_j \partial_{++} X^j) = (2\tilde{\nabla}_j R^i{}_k - P^i{}_l H^l_{jm} R^m{}_k) \psi_+^j \psi_+^k \ , \tag{2.14}$$

where $\tilde{\nabla}$ is defined by

$$\tilde{\nabla}_i := P^j{}_i \nabla_j \ . \tag{2.15}$$

Combining (2.14) with the bosonic boundary condition arising directly from the variation we find

$$\widehat{E}_{ji} = \widehat{E}_{ik} R^k{}_j \ , \tag{2.16}$$

where

$$E_{ij} := g_{ij} + F_{ij} \tag{2.17}$$

[2]The occurrence of (combinations of) field equations as boundary conditions is discussed in [32].

and the hats denote a pull-back to the brane,

$$\widehat{E}_{ij} := \pi^k{}_i \pi^l{}_j E_{kl} \ . \tag{2.18}$$

From (2.18) we find an expression for R,

$$R^i{}_j = (\widehat{E}^{-1})^{ik} \widehat{E}_{jk} - Q^i{}_j \ , \tag{2.19}$$

where the inverse is taken in the tangent space to the brane, i.e.

$$(\widehat{E}^{-1})^{ik} \widehat{E}_{kj} = \pi^i{}_j \ . \tag{2.20}$$

We can multiply equation (2.14) with Q to obtain

$$P^l{}_{[i} P^m{}_{j]} \nabla_l Q^k{}_m = 0 \ . \tag{2.21}$$

Using (2.13) we can show that this implies the integrability condition for π,

$$\pi^l{}_{[i} \pi^m{}_{j]} \nabla_l Q^k{}_m = 0 \ . \tag{2.22}$$

This confirms that the distribution specified by π in TM is integrable and the boundary maps to a submanifold, or brane, B. However, in the Lagrangian approach adopted here, this is implicit in the assumption of Dirichlet boundary conditions. When $F = 0$ the derivative of R along the brane is essentially the second fundamental form, K. Explicitly,

$$K^i_{jk} = P^l{}_j P^m{}_k \nabla_l Q^i{}_m = P^l{}_j \tilde{\nabla}_k Q^i{}_l. \tag{2.23}$$

The left and right supercurrents are

$$T_{+3} := g_{ij} \partial_{++} X^i D_+ X^j - \frac{i}{6} H_{ijk} D_{+3} X^{ijk} \tag{2.24}$$

$$T_{-3} := g_{ij} \partial_{--} X^i D_- X^j + \frac{i}{6} H_{ijk} D_{-3} X^{ijk} \tag{2.25}$$

The conservation conditions are

$$D_- T_{+3} = D_+ T_{-3} = 0 \ . \tag{2.26}$$

The superpartners of the supercurrents are the left and right components of the energy-momentum tensor, $D_+ T_{+3}$ and $D_- T_{-3}$ respectively. If one demands invariance of the total action under supersymmetry one finds that, on the boundary, the currents are related by

$$T_{+3} = \eta T_{-3} \tag{2.27}$$

$$D_+ T_{+3} = D_- T_{-3} \ . \tag{2.28}$$

The supercurrent boundary condition has a three-fermion term which implies the vanishing of the totally antisymmetric part of

$$2Y_{i,jk} + P^l{}_i H_{ljm} R^m{}_k + \frac{1}{6}(H_{ijk} + H_{lmn} R^l{}_i R^m{}_j R^n{}_k) \ , \tag{2.29}$$

where

$$Y_{i,jk} := (R^{-1})_{jl} \tilde{\nabla}_i R^l j \ . \tag{2.30}$$

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3. Additional symmetries

A general variation of (2.1), neglecting boundary terms, gives

$$\begin{aligned} \delta S &= \int dz\, 2g_{ij}\delta X^i \nabla_-^{(+)} D_+ X^j \\ &= -\int dz\, 2g_{ij}\delta X^i g_{ij} \nabla_+^{(-)} D_- X^j \ . \end{aligned} \tag{3.1}$$

The additional symmetries we shall discuss are transformations of the form,

$$\delta_\pm X^i = a^{\pm\ell} L^{(\pm)i}{}_{j_1\dots j_\ell} D_{\pm\ell} X^{j_1\dots j_\ell} \ , \qquad D_{\pm\ell} X^{j_1\dots j_\ell} := D_\pm X^{j_1} \dots D\pm + X^{j_\ell} \ , \tag{3.2}$$

where $L^{(\pm)}$ are vector-valued ℓ-forms such that

$$\lambda^{(\pm)}{}_{i_1\dots i_{\ell+1}} := g_{i_1 j} L^{(\pm)j}{}_{i_2\dots i_{\ell+1}} \tag{3.3}$$

are $(\ell+1)$-forms which are covariantly constant with respect to $\nabla^{(\pm)}$. For example, a left transformation of this type gives

$$\begin{aligned} \delta S &= \int dz\, 2a^{+\ell} \lambda^{(+)}{}_{i_1\dots i_{\ell+1}} D_{+\ell} X^{i_2\dots i_{\ell+1}} \nabla_-^{(+)} D_+ X^{i_1} \\ &= \int dz\, \frac{2}{\ell+1} a^{+\ell} \lambda^{(+)}{}_{i_1\dots i_{\ell+1}} \nabla_-^{(+)} D_{+(\ell+1)} X^{i_1\dots i_{\ell+1}} \\ &= \int dz\, (-1)^\ell D_- \left(\frac{2}{\ell+1} a^{+\ell} \lambda^{(+)}{}_{i_1\dots i_{\ell+1}} D_{+(\ell+1)} X^{i_1\dots i_{\ell+1}} \right) \ , \end{aligned} \tag{3.4}$$

where the last step follows from covariant constancy of $\lambda^{(+)}$ and the chirality of the parameters,

$$D_- a^{+\ell} = D_+ a^{-\ell} = 0 \ . \tag{3.5}$$

Hence these transformations are symmetries of the sigma model without boundary. In the torsion-free case the λs will be the forms which exist on the non-symmetric Riemannian manifolds on Berger's list. There is no such list in the presence of torsion but the same forms will define reductions of the structure group to the various special holonomy groups. In order to preserve the symmetry on the boundary we must have both left and right symmetries so there must be two independent such reductions. Thus we can say that we are interested in boundary sigma models on manifolds which have bi-G-structures.

The λ-forms can be used to construct currents $L^{(\pm)}_{\pm(\ell+1)}$,

$$L^{(\pm)}_{\pm(\ell+1)} := \lambda^{(\pm)}{}_{i_1\dots i_{\ell+1}} D_{\pm(\ell+1)} X^{i_1\dots i_{\ell+1}} \tag{3.6}$$

If we make both left and right transformations of the type (3.2) we obtain

$$\begin{aligned} \delta S &= \frac{2(-1)^\ell}{\ell+1} \int d^2x D_+ D_- \left(D_-(a^{+\ell} L^{(+)}_{\pm(\ell+1)}) - D_+(a^{-\ell} L^-_{-(\ell+1)}) \right) \\ &= \frac{i(-1)^{\ell+1}}{\ell+1} \int_{\partial\Sigma} \left(D_+(a^{+\ell} L^{(+)}_{\pm(\ell+1)}) - D_-(a^{-\ell} L^-_{-(\ell+1)}) \right) \ . \end{aligned} \tag{3.7}$$

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In order for a linear combination of the left and right symmetries to be preserved in the presence of a boundary the parameters should be related by

$$a^{+\ell} = \eta_L a^{-\ell} \ , \tag{3.8}$$

$$D_+ a^{+\ell} = \eta\eta_L D_- a^{-\ell} \ , \tag{3.9}$$

on the boundary, where $\eta_L = \pm 1$.[3] This implies that the currents and their superpartners should satisfy the boundary conditions

$$L^{(+)}_{+(\ell+1)} = \eta\eta_L L^{(-)}_{-(\ell+1)} \ , \tag{3.10}$$

$$D_+ L^{(+)}_{+(\ell+1)} = \eta_L D_- L^{(-)}_{-(\ell+1)} \ . \tag{3.11}$$

The boundary condition (3.10) implies

$$\lambda^{(+)}{}_{i_1 \dots i_{\ell+1}} = \eta_L \eta^\ell \lambda^{(-)}{}_{j_1 \dots j_{\ell+1}} R^{j_1}{}_{i_1} \dots R^{j_{\ell+1}}{}_{i_{\ell+1}} \ . \tag{3.12}$$

The algebra of left (or right) transformations was computed in the torsion-free case in [4]. The commutators involve various generalised Nijenhuis tensors and the classical algebra has a non-linear structure of W-type. In fact, the generalised Nijenhuis tensors vanish in the absence of torsion. However, this is not the case when torsion is present. The commutator of two plus transformations of the type given in (3.2) is (we drop the pluses on the tensors to simplify matters),

$$[\delta_L, d_M] = \delta_P + \delta_N + \delta_K \tag{3.13}$$

where P and N are antisymmetric tensors given by

$$P_{LM} = (L \cdot M)_{[L,M]} := L_{p[L} M^p{}_{M]} \tag{3.14}$$

and

$$N_{iLM} = (\ell + m + 1) H_{jk[i} L^j{}_L M^k{}_{M]} + (-1)^\ell \frac{\ell m}{6} H_{[i\ell_1\ell_2} Q_{L_3 M]} \ . \tag{3.15}$$

The $(\ell + m - 2)$-form Q is defined by

$$Q_{L_2 M_2} = \frac{g^{ij} (L \cdot M)_{[iL_2, |j| M_2]}}{n - (\ell + m - 2)} \ . \tag{3.16}$$

Here L stands for ℓ antisymmetrised indices, L_2 indicates that the first of these should be omitted and so on. Square brackets around the multi-indices indicate antisymmetrisation over all of the indices. The δ_K transformation is generated by the conserved current $K := TQ$, where $Q := Q_{L_2 M_2} D_{+(\ell+m-2)} X^{L_2 M_2}$. Note that P and Q can be zero and that N is not the Nijenhuis concomitant except in the special case that $L = M = J$, an almost complex structure.

The left and right symmetries commute up to the equations of motion. In the case of $(2, 2)$ models, closure of the left and right algebras separately requires the two type

[3] In the case that there is one pair of L tensors.

(1, 1) tensors $J^{(\pm)}$ to be complex structures. They need not commute unless one demands off-shell closure without the introduction of further auxiliary fields. However, any two left and right symmetries of the above type commute up to a generalised commutator term as a simple argument shows. Let $\delta_\pm$ denote left and right variations with two L-tensors, of different rank in general. We have

$$\delta_+\delta_- X^i = \delta_+ \left(a^{-m} L^{(-)i}{}_K D_{-m} X^K \right) , \tag{3.17}$$

where K denotes a multi-index with m antisymmetrised indices. Since all of the K indices are contracted we can replace the δ_+ variation by a covariant variation with the Levi-Civita connection provided that we take care of the remaining i index. The explicit connection term drops out in the commutator by symmetry. In the remaining terms one can introduce either $\nabla^{(-)}$, acting on $L^{(-)}$, or $\nabla^{(+)}$, acting on $\delta_+ X^k$, and then show that all of the torsion terms cancel, bar one, again coming from the i index. However, this cancels in the commutator too, because the plus and minus connections are swapped in the other term. One thus finds

$$\begin{aligned}[][\delta_+, \delta_-] X^i =& (-1)^n mn\, a^{-m} a^{+n} \left(L^{(-)i}{}_{mK_2} L^{(+)m}{}_{pL_2} - L^{(+)i}{}_{mL_2} L^{(-)m}{}_{pL_2} \right) \times \\ & \left(D_{+(l-1)} X^{L_2} D_{-(m-1)} X^{K_2} \right) \times \left(\nabla_-^{(+)} D_+ X^p \right) ,\end{aligned} \tag{3.18}$$

the third factor being the equation of motion. The multi-index L associated with $L^{(+)}$ stands for n antisymmetrised indices.

4. Consistency

In this section we shall examine the consistency of the boundary conditions, i.e we investigate the orbits of the boundary conditions under symmetry variations to see if further constraints arise. We shall show that the supersymmetry variation of the L-boundary condition (3.10) and the L-variation of the fermion boundary condition (2.8) are automatically satisfied if (3.12) is. To see this we differentiate (3.12) along B to obtain

$$Y^{(+)}{}_{k,[i_1}{}^m \lambda^{(+)}{}_{i_2 \ldots i_{\ell+1}]m} = 0 , \tag{4.1}$$

where

$$Y^{(+)}{}_{i,jk} := (R^{-1})_{jl} (\tilde{\nabla}_i^{(+)} R^l{}_k - H^l{}_{im} R^m{}_k) . \tag{4.2}$$

Note that we have contracted the derivative with P rather than π; this is permissible due to the fact that $P\pi = \pi P = P$. Equation (4.1) says that $Y^{(+)}$, regarded as a matrix-valued one-form, takes its values in the Lie algebra of the group which leaves the form $\lambda^{(+)}$ invariant. The constraint corresponding to the superpartner of the L-current boundary condition is just the totally antisymmetric part of (4.1).

We now consider the variation of the fermionic boundary condition under L-transformations. We need to make both left and right transformations which together can be written

$$\delta X^i = 2a^{+\ell} P^i{}_k L^{(+)k}{}_{j_1 \ldots j_\ell} D_{+\ell} X^{j_1 \ldots j_\ell} \tag{4.3}$$

$$= 2a^{-\ell} P^i{}_k L^{(-)k}{}_{j_1 \ldots j_\ell} D_{-\ell} X^{j_1 \ldots j_\ell} \ . \tag{4.4}$$

A straightforward computation yields

$$(2\tilde{\nabla}_{[k} R^i{}_{m]} - P^i{}_n H^n{}_{[k|p|} R^p{}_{m]}) L^{(+)k}{}_{j_1 \ldots j_\ell} D_{+(\ell+1)} X^{j_1 \ldots j_\ell m} = 0 \ . \tag{4.5}$$

We define

$$Z^{(+)}_{i,jk} = (R^{-1})_{il} (2\tilde{\nabla}_{[j} R^l{}_{k]} + P^l{}_m H^m{}_{n[j} R^n{}_{k]}) \ , \tag{4.6}$$

which is the term in the bracket in (4.5) multiplied by R^{-1}. We claim that

$$Y^{(+)}_{i,jk} = Z^{(+)}_{i,jk} \ . \tag{4.7}$$

This can be proved using (2.29) with the aid of a little algebra. Thus we have shown that, if the boundary conditions (3.12) are consistent, then the constraints following from supersymmetry variations of the L-constraints and from L-variations of the fermionic boundary condition are guaranteed to be satisfied.

If $\lambda^{(+)} = \lambda^{(-)} := \lambda$ the boundary condition (3.12) typically implies that $\pm R$ is an element of the group which preserves λ. If this is the case, then (4.1) becomes an identity. However, it can happen that R is not an element of the invariance group but that $R^{-1}dR$ still takes its values in the corresponding Lie algebra. For example, if λ is the two-form of a $2m$-dimensional Kähler manifold and the sign $\eta_L \eta = -1$, R is not an element of the unitary group but, since it must have mixed indices, it is easy to see that $R^{-1}dR$ is itself $\mathfrak{u}(m)$-valued.

A similar argument applies in the general case, when $\lambda^{(+)} \neq \lambda^{(-)}$. In the next section we discuss how the plus and minus forms are related by an element V of the orthogonal group (see (5.5)). Thus equation (3.12) can be written

$$\lambda^{(-)}{}_{i_1 \ldots i_{\ell+1}} = \eta_L \eta^\ell \lambda^{(-)}{}_{j_1 \ldots j_{\ell+1}} \widehat{R}^{j_1}{}_{i_1} \ldots \widehat{R}^{j_{\ell+1}}{}_{i_{\ell+1}} \ , \tag{4.8}$$

where $\widehat{R} := RV^{-1}$. If we differentiate (4.8) along the brane with respect to the minus connection we can then use the above argument applied to $\widehat{R}$.

5. Target space geometry

In this section we discuss the geometry of the sigma model target space in the presence of torsion when the holonomy groups of the torsion-full connections $\nabla^{(\pm)}$ are of special type, specifically G_2, $Spin(7)$ and $SU(3)$. We use only the data given by the sigma model and use a simple approach based on the fact that there is a transformation which takes one from one structure to the other. We begin with G_2 and then derive the other two cases from this by dimensional reduction and oxidation.

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G_2. In this case we have a seven-dimensional Riemannian manifold (M, g) with two G_2-forms $\varphi^{(\pm)}$ which are covariantly constant with respect to left and right metric connections $\nabla^{(\pm)}$ such that the torsion tensor is $\pm H$. G_2 manifolds with torsion have been studied in the mathematical literature [33, 34] and have arisen in supergravity solutions [27]. Bi-G_2-structures have also appeared in this context and have been given an interpretation in terms of generalised G_2-structures [21]. They can be studied in terms of a pair of covariantly constant spinors from which one can construct the G_2-forms, as well as other forms, as bilinears. We will not make use of this approach here, preferring to use the tensors given to us naturally by the sigma model. As noted in [27] there is a common $SU(3)$ structure associated with the additional forms. We shall derive this from a slightly different perspective here.

In most of the literature use is made of the dilatino Killing spinor equation which restricts the form of H. The classical sigma model does not appear to require this restriction as the dilaton does not appear until the one-loop level. The dilatino equation is needed in order to check that one has supersymmetric supergravity solutions but is not essential for our current purposes.

For G_2 there are two covariantly constant forms, the three-form φ and its dual four-form $*\varphi$ (we shall drop the star when using indices). The metric can be written in terms of them. A convenient choice for φ is

$$\varphi = \frac{1}{3!}\varphi_{ijk}e^{ijk} = e^{123} - e^1(e^{47} + e^{56}) + e^2(e^{46} - e^{57}) - e^3(e^{45} + e^{67}) \tag{5.1}$$

This form is valid in flat space or in an orthonormal basis, the e^is being basis forms. Another useful way of think about the G_2 three-form is to write it in a $6 + 1$ split. We then have

$$\begin{aligned} \varphi_{ijk} &= \lambda_{ijk} \\ \varphi_{ij7} &= \omega_{ij} \\ \varphi_{ijk7} &= -\widehat{\lambda}_{ijk}\ , \end{aligned} \tag{5.2}$$

where $i, j, k = 1 \dots 6$, and $\{\lambda, \widehat{\lambda}, \omega\}$ are the forms defining an $SU(3)$ structure in six dimensions. The three-forms λ and $\widehat{\lambda}$ are the real and imaginary parts respectively of a complex three-form Ω which is of type $(3, 0)$ with respect to the almost complex structure defined by ω.

On a G_2 manifold with skew-symmetric torsion, the latter is uniquely determined in terms of the Levi-Civita covariant derivative of φ [33, 34]. This follows from the covariant constancy of φ with respect to the torsion-full connection.

Now suppose we have a bi-G_2-structure. The two G_2 three-forms are related to one another by an $SO(7)$ transformation, V. If we start from $\varphi^{(-)}$ this will be determined up to an element of $G_2^{(-)}$. So we can choose a representative to be generated by an element $w \in \mathfrak{so}(7)$ of the coset algebra with respect to $\mathfrak{g}_2^{(-)}$. This can be written

$$w_{ij} = \varphi^{(-)}_{ijk}v^k \tag{5.3}$$

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and $V = e^w$. The vector v will be specified by a unit vector N and an angle α. It is straightforward to find V,

$$V^i{}_j = \cos\alpha\delta^i{}_j + (1-\cos\alpha)N^iN_j + \sin\alpha\,\varphi^{(-)i}{}_{jk}N^k\ . \tag{5.4}$$

Using

$$\varphi^{(+)} = \varphi^{(-)}V^3\ , \tag{5.5}$$

where one factor of V acts on each of the three indices of φ, we can find the relation between the two G_2 forms explicitly,

$$\varphi^{(+)}_{ijk} = A\varphi^{(-)}_{ijk} + B\varphi^{(-)}_{ijkl}N^l + 3C\varphi^{(-)}_{[ij}{}^lN_{k]}N_l\ , \tag{5.6}$$

where

$$A = \cos 3\alpha, \qquad B = \sin 3\alpha, \qquad C = 1-\cos 3\alpha\ . \tag{5.7}$$

The dual four-forms are related by

$$\varphi^{(+)}_{ijkl} = (A+C)\varphi^{(-)}_{ijkl} - 4B\varphi^{(-)}_{[ijk}N_{l]} - 4C\varphi^{(-)}_{[ijk}{}^mN_{l]}N_m\ . \tag{5.8}$$

The angle α is related to the angle between the two covariantly constant spinors. To simplify life a little we shall follow [27] and choose these spinors to be orthogonal which amounts to setting $\cos\frac{\alpha}{2} = 0$. We then find

$$\varphi^{(+)}_{ijk} = -\varphi^{(-)}_{ijk} + 6\varphi^{(-)}_{[ij}{}^lN_{k]}N_l\ . \tag{5.9}$$

and

$$\varphi^{(+)}_{ijkl} = \varphi^{(-)}_{ijkl} - 8\varphi^{(-)}_{[ijk}{}^mN_{l]}N_m\ . \tag{5.10}$$

We can use the vector N to define an $SU(3)$ structure as above. We set

$$\omega = i_N\varphi^{(-)}\,; \qquad \lambda = \varphi^{(-)} - \omega\wedge N\,; \qquad \widehat{\lambda} = i_N * \varphi^{(-)}\ . \tag{5.11}$$

The three-form $\widehat{\lambda}$ is the six-dimensional dual of λ and the set of forms $\{\omega,\lambda,\widehat{\lambda}\}$ is the usual set of forms associated with an $SU(3)$ structure in six dimensions. For the plus forms we have

$$\begin{aligned} i_N\varphi^{(+)} &= \omega \\ \varphi^{(+)} - \omega\wedge N &= -\lambda \\ i_N * \varphi^{(+)} &= -\widehat{\lambda}\ . \end{aligned} \tag{5.12}$$

Thus a bi-G_2-structure is equivalent to a single G_2 structure together with a unit vector (and an angle to be more general). The unit vector N then allows one to define a set of $SU(3)$ forms as above. In [27] it is shown that the projector onto the six-dimensional subspace is integrable, but this presupposes that the dilatino Killing spinor equation holds. Since we make no use of this equation in this paper it need not be the case that integrability holds.

It is straightforward to construct a covariant derivative $\widehat{\nabla}$ which preserves both G_2 structures. This connection has torsion but this is no longer totally antisymmetric. It is enough to show that the covariant derivatives of N and $\varphi^{(-)}$ are both zero. If we write

$$\widehat{\nabla}_i N_j = \nabla_i^{(-)} N_j - S_{i,j}{}^k N_k \ , \tag{5.13}$$

where $S_{i,jk} = -S_{i,kj}$, then these conditions are fulfilled if

$$S_{i,jk} = \frac{1}{2}H_{ijk} + \frac{1}{4}H_i{}^{lm}\varphi^{(-)}_{lmjk} - \frac{3}{2}H_{ilm}\Pi^l{}_j\Pi^m{}_k - \frac{3}{4}H_i{}^{lm}\varphi^m{}_{ijkln}N^n N_m \ . \tag{5.14}$$

Here $\Pi^i{}_j := \delta^i{}_j - N^i N_j$ is the projector transverse to N.

$SU(3)$**.** Manifolds with $SU(3) \times SU(3)$ have arisen in recent studies of supergravity solutions with flux [28]–[30]. They have also been discussed in a recent paper on generalised calibrations [25]. A bi-$SU(3)$ structure on a six-dimensional manifold is given by a pair of a pair of forms $\{\omega^{(\pm)}, \Omega^{(\pm)}\}$ of the above type which are compatible with the metric. If the sigma model algebra closes off-shell the complex structures will be integrable. The transformation relating the two structures can be found using a similar construction to that used in the G_2 case. However, we can instead derive the relations between the plus and minus forms by dimensional reduction from G_2. To this end we introduce a unit vector N', which we can take to be in the seventh direction, and define the $SU(3)$ forms as in equation (5.2) above. We consider only the simplified bi-G_2-structure and we also then take the unit vector N to lie within the six-dimensional space. The unit vector N now defines an $SO(6)$ transformation. The relations between the plus and minus forms are given by

$$\begin{aligned}
\omega^{(+)}_{ij} &= -\omega^{(-)}_{ij} + 4\omega^{(-)k}_{[i} N_{j]} N_k \\
\lambda^{(+)}_{ijk} &= -\lambda^{(-)}_{ijk} + 6\lambda_{[ij}{}^l N_{k]} N_l . \\
\widehat{\lambda}^{(+)}_{ijk} &= \widehat{\lambda}^{(-)}_{ijk} - 6\widehat{\lambda}^{(-)l}_{[ij} N_{k]} N_l \ .
\end{aligned} \tag{5.15}$$

We can rewrite this in complex notation if we introduce the three-forms $\Omega^{(\pm)} := \lambda^{(\pm)} + i\widehat{\lambda}^{(\pm)}$ and split N into $(1,0)$ and $(0,1)$ parts, $n, \bar{n}$. So

$$N_i = n_i + \bar{n}_i \, ; \qquad i\omega_{ij} N^j = n_i - \bar{n}_i \ . \tag{5.16}$$

Note that $n \cdot \bar{n} = \frac{1}{2}$. Then equations (5.15) are equivalent to

$$\begin{aligned}
\omega^{(+)}_{ij} &= -\omega^{(-)}_{ij} - 2in_{[i}\bar{n}_{j]} \\
\Omega^{(+)}_{ijk} &= 6\bar{\Omega}^{(-)l}_{[ij} n_{k]} n_l \ .
\end{aligned} \tag{5.17}$$

This type of bi-$SU(3)$-structure is therefore equivalent to a single $SU(3)$ structure together with a normalised $(1,0)$-form.

Spin(7). A *Spin*(7) structure on an eight-dimensional Riemannian manifold is specified by a self-dual four-form Φ of a certain type. In a given basis its components can be constructed from those of the G_2 three-form. Thus

$$\Phi_{abcd} = \varphi_{abcd}\,; \qquad \text{and} \qquad \Phi_{abc8} = \varphi_{abc}\ , \tag{5.18}$$

where, in this section, $a, b, \ldots$ run from 1 to 7 and $i, j, \ldots$ run from 1 to 8. *Spin*(7) geometry with skew-symmetric torsion has been discussed [33, 35] and generalised *Spin*(7) structures have also been studied [21]. A bi-*Spin*(7)-structure on a Riemannian manifold consists of a pair of such forms, covariantly constant with respect to $\nabla^{(\pm)}$. We can again get from the minus form to the plus form by an orthogonal transformation, but since the dimension of $SO(8)$ minus the dimension of *Spin*(7) is seven it is described by seven parameters. In the presence of a *Spin*(7) structure one of the chiral spinor spaces, Δ_s, say, splits into one- and seven-dimensional subspaces, $\Delta_s = \mathbb{R} \oplus \Delta_7$. The transformation we seek will be described by a unit vector $n^a \in \Delta_7$ together with an angle.

It will be useful to introduce some invariant tensors for *Spin*(7) using this decomposition of the spin space. We set

$$\phi_{ajk} = \begin{cases} \phi_{abc} = \varphi_{abc} \\ \phi_{ab8} = \delta_{ab} \end{cases} \tag{5.19}$$

$$\phi_{abkl} = \begin{cases} \phi_{ab}{}^{cd} = \varphi_{ab}{}^{cd} - 2\delta^{cd}_{[ab]} \\ \phi_{abc8} = -\varphi_{abc} \end{cases}\ , \tag{5.20}$$

where φ_{abc} is the G_2 invariant. It will also be useful to define

$$\phi_{aijkl} := \phi_{ab[ij}\phi^b{}_{kl]}\ . \tag{5.21}$$

The *Spin*(7) form itself can be written as

$$\Phi_{ijkl} = \phi_{a[ij}\phi^a{}_{kl]}\ . \tag{5.22}$$

The space of two-forms splits into $7 + 21$, and one can project onto the seven-dimensional subspace by means of ϕ_{ajk}. With these definitions we can now oxidise the G_2 equations relating the plus and minus structure forms to obtain

$$\Phi^{(+)}_{ijkl} = -\Phi^{(-)}_{ijkl} - 6 n_a n_b \phi^{(-)a}{}_{[ij}\phi^{(-)b}{}_{kl]}\ . \tag{5.23}$$

Here the unit vector N in the G_2 case becomes the unit spinor n.

6. Examples of solutions

In this section we look at solutions to the boundary conditions for the additional symmetries which can be identified with various types of brane. We shall go briefly through the main examples, confining ourselves to $U(\frac{n}{2})$, $SU(\frac{n}{2})$ and the exceptional cases G_2 and *Spin*(7).

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$U(\frac{n}{2}) \equiv U(m)$. This case corresponds to $N = 2$ supersymmetry. For $H = F = 0$ we assume that the supersymmetry algebra closes off-shell so that M is a Kähler manifold with complex structure J, hermitian metric g and Kähler form ω. The Kähler form is closed and covariantly constant. The boundary conditions for the second supersymmetry, which can be viewed as an additional symmetry with $\lambda = \omega$, imply

$$\omega_{ij} = \pm\omega_{kl} R^k{}_i R^l{}_j \; . \tag{6.1}$$

Thus there are two possibilities, type A where $JR = -RJ$ and type B where $JR = RJ$ [36]. Consider type B first. In this case the brane inherits a Kähler structure from the target space and so has dimension $2k$. If there is a non-vanishing gauge field F, it must be of type $(1,1)$ with respect to this structure. The calibration form is ω^k.

For type B with zero F field, J is off-diagonal in the orthonormal basis in which R takes its canonical form

$$R = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} , \tag{6.2}$$

where p and q denote the dimensions of B and the transverse tangent space, $p + q = n$, and 1_p, 1_q denote the corresponding unit matrices. The only possibility is $p = q = m$. The Kähler form vanishes on both the tangent and normal bundles to the brane, so that the brane is Lagrangian.

When the F field is non-zero the situation is more complicated. We may take R to have the same block-diagonal form as in (6.2) but with 1_p replaced by R_p. From (2.19)

$$R_p = (1 + F)^{-1}(1 - F) \, . \tag{6.3}$$

The analysis of $JR = -RJ$ shows that the brane is coisotropic [37]. This means that there is a $4k$-dimensional subspace in each tangent space to the brane where J is non-singular, there is an r-dimensional subspace on which it vanishes, and the dimension of the normal bundle is also r. The product (J_pF) is an almost complex structure and both J_p and F are of type $(2,0) + (0,2)$ with respect to (J_pF). For $m = 3$ we can therefore only have $p = 5$. For $m = 4$ we can have $p = 5$ but we can also have a space-filling brane with $p = 8$.

$N = 2$ sigma models with boundary and torsion have been discussed in [10]; the geometry associated with the boundary conditions is related to generalised complex geometry [24, 23].

$SU(\frac{n}{2}) \equiv SU(m)$. In the Calabi-Yau case we have, in addition to the Kähler structure, a covariantly constant holomorphic $(m,0)$ form Ω where $m = \frac{n}{2}$. There are two independent real covariantly constant forms, λ and $\widehat{\lambda}$, which can be taken to be the real and imaginary parts of Ω. The corresponding L-tensors which define the symmetry transformations are related by

$$\widehat{L}^i{}_{j_1 \ldots j_{m-1}} = J^i{}_k L^k{}_{j_1 \ldots j_{m-1}} \tag{6.4}$$

Because there are now two currents we can introduce a phase rather than a sign in the boundary condition. Thus

$$\Omega_{i_1 \ldots i_m} = e^{i\alpha} \Omega_{j_1 \ldots j_m} R^{j_1}{}_{i_1} \ldots R^{j_m}{}_{i_m} \tag{6.5}$$

A second possibility is that Ω on the right-hand side is replaced by $\bar{\Omega}$. For type B branes, the displayed equation is the correct condition. The R-matrix is the sum of holomorphic and anti-holomorphic parts, $R = \mathcal{R} \oplus \bar{\mathcal{R}}$, and (6.5) implies that

$$\det \mathcal{R} = e^{i\alpha} \; . \tag{6.6}$$

If $F = 0$ this fixes the phase, but if $F \neq 0$ it imposes a constraint on F which must in any case be a $(1,1)$ form (from $JR = RJ$) [38]. The constraint is

$$\det \mathcal{R}_p = e^{i\alpha}(-1)^{\frac{q}{2}} \; , \tag{6.7}$$

or

$$\det(1+f) = e^{i\alpha}(-1)^{\frac{q}{2}} \det(1-f) \tag{6.8}$$

where $f^a{}_b = g^{a\bar{c}} F_{\bar{c}b}$, in a unitary basis.

For type A branes, from $JR = -RJ$ it follows that R maps holomorphic vectors to anti-holomorphic ones and vice versa so that $\bar{\Omega}$ must be used in (6.5). In the case that $F = 0$ the brane is a SLAG with $\mathfrak{Re}\Omega$ as the calibration form. For $F \neq 0$ we have coisotropic branes with an additional constraint on the gauge field [38].

The geometry of the bi-$SU(m)$ case has been studied from the point of view of generalised geometry and generalised calibrations in [25].

G_2. The boundary conditions associated with the G_2 currents are

$$\begin{aligned} \varphi &= \eta_L \varphi R^3 \\ *\varphi &= \eta_L * \varphi R^4 \det R \end{aligned} \tag{6.9}$$

We consider first $F = H = 0$. From the first of these equations it follows that $(\eta_L R) \in G_2$. From this it follows that the sign in the boundary condition for $*\varphi$ is always positive because the sign of the determinant of R is equal to η_L. Thus the second constraint reduces to $*\varphi = *\varphi R^4$.

There are two possibilities depending on the sign of η_L. If it is positive then non-zero components of φ must have an even number of normal indices, whereas if it is negative they must have an odd number of non-zero components. Since $(\eta_L R) \in G_2$, and is symmetric, it can be diagonalised by a G_2 matrix so that we can bring R to its canonical form in a G_2 basis. Looking at the components of φ we see that the only possibilities which are compatible with the preservation of the non-linear symmetries on the boundary are either $\eta_L = 1$, in which case B is a three-dimensional associative cycle, or $\eta_L = -1$ in which case B is a four-dimensional co-associative cycle [13].

Now let us turn to $F \neq 0$, but $H = 0$. We shall assume that the tangent bundle M, restricted to the brane, splits into three, $TM|_B = T_1 \oplus T_2 \oplus N$, of dimensions p_1, p_2 and q respectively. N is the normal bundle and $R|_{T_2} = 1_{p_2}$. If there is at least one normal direction we may assume that one of these is 7 in the conventions of (5.2). Thus the problem is reduced to a six-dimensional one, at least algebraically. The six-dimensional boundary conditions are (where R is now a 6×6 matrix),

$$\begin{aligned} \lambda &= \eta_L \lambda R^3 \\ \widehat{\lambda} &= \eta_L \widehat{\lambda} R^3 \det R \\ \omega &= -\eta_L \omega R^2 \ , \end{aligned} \tag{6.10}$$

in an obvious notation. If the sign is negative the brane is type B, whereas if $\eta_L = +1$ we have type A. These are the same conditions as we have just discussed in the preceding section, the only difference being that the phase is not arbitrary. The constraints on the F field are therefore slightly stronger.

The last possibility is a space-filling brane in seven dimensions. Since F is antisymmetric there must be at least one trivial direction for R so that we can again reduce the algebra to the six-dimensional case. The only possibilty is $\eta = +1$ in which case we have type B. The non-trivial dimension must be even, and since $\det \mathcal{R} = 1$ the case $p = 2$ is also trivial.

Now let us consider the case with torsion. The boundary condition for the non-linear symmetries associated with the forms yield

$$\begin{aligned} \varphi^{(+)} &= \eta_L \varphi^{(-)} R^3 \\ *\varphi^{(+)} &= \eta_L * \varphi^{(-)} R^4 \det R \ , \end{aligned} \tag{6.11}$$

When the brane is normal to N we find, on the six-dimensional subspace,

$$\begin{aligned} \lambda &= -\eta \lambda R^3 \\ \widehat{\lambda} &= -\eta \widehat{\lambda} R^3 \det R \\ \omega &= -\eta \omega R^2 \ , \end{aligned} \tag{6.12}$$

The analysis is very similar to the case of zero torsion with F. One finds that $\eta_L = -1$ corresponds to type B while $\eta_L = +1$ is type A. In particular, for type B there is a five-brane which corresponds to the five-brane wrapped on a three-cycle discussed in the supergravity literature [26, 27].

Spin(7)**.** In the absence of torsion, the boundary condition associated with the conserved current is

$$\Phi = \widehat{\eta} \Phi R^4 \ , \tag{6.13}$$

for some sign factor $\widehat{\eta}$. If this is negative then $\det R$ is also negative so that the dimension of B must be odd. Furthermore, Φ must have an odd number of normal indices with respect to the decomposition of the tangent space induced by the brane. However, one can show that such a decomposition is not compatible with the algebraic properties of Φ. Therefore the sign $\widehat{\eta}$ must be positive. It is easy to see that a four-dimensional B is compatible with this, and indeed we then have the standard Cayley calibration with Φ pulled-back to the brane being equal to the induced volume form. On the other hand if B has either two or six dimensions one can show that it is not compatible with the *Spin*(7) structure.

As one would expect, therefore, the only brane compatible with the non-linear symmetry associated with Φ on the boundary is the Cayley cycle [13].

If $F \neq 0$, but $H = 0$, and if we assume that there is at least one direction normal to the brane, then the $Spin(7)$ case reduces to G_2 (with $F \neq 0$). If the brane is space-filling but there is at least one trivial direction, then there must be at least two by symmetry and again we recover the G_2 case. But we can also have a space-filling brane which is non-trivial in all eight directions.

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